Simultaneous Exact/Approximate Boundary Controllability of Thermo-Elastic Plates with Variable Thermal Coefficient and Moment Control

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We study a controllability problem (exact in the mechanical variables \(u, v\) and, simultaneously, approximate in the thermal variable \(\theta\)) of thermo-elastic plates by means of boundary controls, in the hinged/Dirichlet BC case, when the “thermal expansion” term is variable in space.

1. THERMO-ELASTIC SYSTEMS: BOUNDARY CONTROLLABILITY PROBLEM

Let \(\Omega \subset \mathbb{R}^2\) be an open bounded domain with smooth boundary \(\Gamma\). We shall here consider the following thermo-elastic plate [La.1, L-L.1] on a finite time interval in the unknown \(u(t, x)\) (vertical displacement) and

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\[ \theta(t, x) \text{ (relative temperature about the stress-free state } \theta = 0), \]
\[
\begin{align*}
\theta_{tt} - \gamma \Delta \theta_{tt} + \Delta^2 \theta + \text{div}(\alpha(x) \nabla \theta) &= 0 \quad \text{in } Q \equiv (0, T) \times \Omega, \\
\theta_t - \Delta \theta - \text{div}(\alpha(x) \nabla w) &= 0 \quad \text{in } Q, \\
w(0, \cdot) &= w_0; \quad w_t(0, \cdot) = \theta_0; \quad \theta(0, \cdot) = \theta_0 \quad \text{in } \Omega,
\end{align*}
\]
(1.1a)
\[
\begin{align*}
\theta_{t} - \Delta \theta - \text{div}(\alpha(x) \nabla w) &= 0 \quad \text{in } Q, \\
w(0, \cdot) &= w_0; \quad w_t(0, \cdot) = \theta_0; \quad \theta(0, \cdot) = \theta_0 \quad \text{in } \Omega,
\end{align*}
\]
(1.1b)
\[
\begin{align*}
w(0) &= w_{0}; \quad w_t(0) = \theta_1; \quad \theta(0) = \theta_0 \quad \text{in } \Sigma = (0, T) \times \Gamma.
\end{align*}
\]
(1.1c)

to be augmented by boundary conditions on \( \partial \Omega = \Gamma \). Throughout this paper the constant \( \gamma \) is positive, \( \gamma > 0 \), so that the model accounts for moments of inertia (rotational forces). The coefficient \( \gamma \) is proportional to the square of the thickness of the plate. The thermal coefficient \( \alpha(x) \) represents “thermal expansion” of the plate material and is assumed to be a function of \( x \in \Omega \) and of class \( C^2(\Omega) \). In this paper, we shall consider the case of boundary controls associated with system (1.1) in the hinged mechanical/Dirichlet thermal boundary conditions (BC):
\[
\begin{align*}
w_{|\Sigma} &= u_1, \quad \Delta w_{|\Sigma} = u_2, \quad \theta_{|\Sigma} = u_3 \quad \text{on } \Sigma = (0, T) \times \Gamma.
\end{align*}
\]
(1.2)

Well-posedness of the mixed problems above, (1.1), (1.2), is discussed in [Tr.2]. (The analysis of the paper works verbatim on \( \mathbb{R}^n \), for any \( n \geq 2 \).)

**Boundary Controllability Problem**

Qualitatively, the boundary controllability problem studied in this note is as follows. Let \( T > 0 \) be sufficiently large, depending on the geometry of \( \Omega \). Given any initial condition \( \{w_0, w_1, \theta_0\} \) and any preassigned target condition \( \{w_0, w_1, \theta_1\} \) in specified Sobolev spaces, seek boundary controls \( \{u_1, u_2, u_3\} \) in specified function spaces (compatible with the regularity of the underlying dynamics) that steer the solution of the corresponding mixed problem (1.1), (1.2) to a state \( \{w(T), w_1(T), \theta(T)\} \) at time \( T \), such that: \( w(T) = w_{0, T}, \quad w_1(T) = w_{1, T}, \quad \text{while } \theta(T) \text{ is arbitrarily “close” to } \theta_1 \text{ in the relevant topology. Thus, the above is a problem of exact controllability in the mechanical variable and, simultaneously, of approximate controllability in the thermal variable. A more precise statement is given in the following theorem.}

**Main Result**

Our main result on the above boundary exact/approximate controllability problem follows next.

**Theorem 1.1.** Let \( \Gamma_1, \Gamma_2, \Gamma_3 \subset \Gamma \) be open subsets of the boundary \( \Gamma \), with a non-empty intersection of positive measure. (We think of \( \Gamma_1 \) and \( \Gamma_3 \) as being arbitrarily small.) Moreover, regarding \( \Gamma_2 \), we assume that: there exists a point \( x_0 \in \mathbb{R}^2 \), such that
\[
(x - x_0) \cdot \nu(x) \leq 0 \quad \text{for } x \in \Gamma \setminus \Gamma_2,
\]
(1.3)
where here and throughout the paper $v(x)$ denotes the unit outward normal at $x \in \Gamma$. Let

$$T_0 \equiv 2\sqrt{\gamma} \max \sup_{x \in \Omega} \text{dist}(x, \Gamma)_{i}, \quad i = 1, 2, 3. \quad (1.4)$$

Let $\alpha \in C^2(\Omega)$. Finally, let $\{w_0, w_1, \theta_0\}$ and $\{w_{0,T}, w_{1,T}, \theta_T\}$ be pre-assigned initial and target states, with

$$\{w_0, w_1\} \quad \text{and} \quad \{w_{0,T}, w_{1,T}\} \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega); \quad (1.5)$$

$$\theta_0, \theta_T \in H^1_0(\Omega).$$

Then, for any $T > T_0$ and any $\delta > 0$ arbitrarily small, there exist boundary control functions

$$u_1 = \begin{cases} \hat{u}_1 \in C^\infty_0(\Sigma_1) & \text{on } \Sigma - \Sigma_1, \\ 0 & \text{on } \Sigma - \Sigma_1, \end{cases} \quad u_2 = \begin{cases} \hat{u}_2 \in L^2_2(\Sigma_2) & \text{on } \Sigma - \Sigma_2, \\ 0 & \text{on } \Sigma - \Sigma_2, \end{cases} \quad u_3 = \begin{cases} \hat{u}_3 \in C^\infty_0(\Sigma_3) & \text{on } \Sigma - \Sigma_3 \end{cases} \quad (1.6)$$

(in particular, with all time derivatives $u_1^{(n)}(0) = u_3^{(n)}(T) = 0, i = 1, 3,$ and all $n = 0, 1, 2,$ vanishing at $t = 0,$ and $t = T$), such that the corresponding solution of the mixed problem (1.1), (1.2) satisfies the following terminal condition at $T$:

$$w(T) = w_{0,T}, \quad w_1(T) = w_{1,T}, \quad \|\theta(T) - \theta_T\|_{H^1(\Omega)} \leq \delta. \quad (1.7)$$

Remark 1.1. We note that in the above theorem the essential, critical control mechanism is provided by the control $u_2$ in the highest mechanical BC: with $u_2 \in L^2_2(\Sigma_2)$ on a suitable portion of the boundary (see (1.6)) in the hinged case. The addition of infinitely smooth controllers $u_1$ and $u_3$ in the lowest mechanical BC and in the thermal BC, compactly supported on $\Sigma_1 = (0, T) \times \Gamma_1$ and $\Sigma_3 = (0, T) \times \Gamma_3$, with $\Gamma_1, \Gamma_3$ arbitrarily small open portions of the boundary having non-empty intersection with $\Gamma_2$, is only for the purpose of obtaining the property of “approximate controllability” of the overall thermo-elastic plate. By duality (Hahn–Banach theorem), this latter property is equivalent to the property of unique continuation across the boundary of a corresponding over-determined dual or adjoint problem, see Theorem 4.2.2 below. At present, the results of this paper, based on [E-L-T2, E-L-T3], which improve over the literature [I.1], require that the dual problem be over-determined with all four boundary conditions on a common, non-empty open portion of the boundary of positive measure, in order to assert that, then, the corresponding solution is identically zero. See the statement of Theorem 4.2.2. This is the reason why we assume three active controls in (1.2), in lieu of just $u_3$. However, any progress in the area of the unique continuation property for thermo-elastic plates will imply corresponding improvements of our results, by allowing us to drop unnecessary controls, such as, possibly, $u_1$ and $u_3$. 
We explicitly remark that since we think of $\Gamma_1$ and $\Gamma_3$ as arbitrarily small, with $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq \emptyset$, we may always assume without loss of generality that $\emptyset \neq \tilde{\Gamma} \equiv \Gamma_1 = \Gamma_3 \subset \Gamma_2$.

Remark 1.2. Many, in fact most, steps in the proof of this paper continue to hold true for a much more general thermo-elastic model than (1.1), where now $(-\Delta)$ is replaced by a space variable coefficient elliptic operator of order two, which is a positive, self-adjoint operator. These steps include: abstract models, the backward uniqueness property quoted from [L-R-T.1], and the critical estimates for wave equations used in Section 6 on Eq. (6.10), where the required estimates would follow from [L-T-Y.2], etc. However, to date, we have not checked in detail if the entire proof of the present paper carries over to the fully space variable coefficient case.

Literature

The problem of controlling thermo-elastic plates has already received attention in the literature [La.1], [La.2], [L–L.1, Chap. 6]. Since a thermo-elastic plate consists of a Kirchoff plate coupled with a heat equation, it is natural to view controllability of a thermo-elastic plate as a “perturbation” of the controllability of an elastic Kirchoff plate. In fact, this strategy works well in the case of distributed (internal) control, where the control operator is bounded on the basic state space; or else when the coefficient $\alpha$ is constant and “suitably small.” In both cases, a classical approach (with roots in finite-dimensional theory) can be used to make the thermo-elastic plate inherit controllability properties from the elastic Kirchoff plate [T-Z.1]. Here, the distributed control acts on a layer of the boundary in the mechanical Kirchoff equation and yields exact controllability in the mechanical variables and, simultaneously approximate controllability in the thermal variable in the constant coefficient case. A stronger result—exact controllability for both the mechanical and thermal variables—is obtained by [Av.1], where, moreover, the distributed control acts in the thermal equation (only), still in the constant coefficient case.

The situation is quite different in the case of boundary controls, where then the control operator is highly unbounded, unless the constant coefficient $\alpha$ is suitably small, in which case a direct perturbation argument over the Kirchoff elastic case works, at least if one is interested only in the exact controllability of the mechanical variables $\{w, w_t\}$, with no regard or information whatsoever to the thermal component. This is the case of [La.2] and [L-L.1], under free, respectively, clamped boundary controls. Smallness of the constant coefficient $\alpha$ plays a critical role in the corresponding perturbation argument in [La.2] and [L-L.1] over the corresponding Kirchoff (elastic) case.
For a thermo-elastic wave (rather than plate), Liu [Liu.1] originally claimed a result of partial exact controllability, with no assumption of smallness on the coupling parameter. However, this claim was later retracted [Liu.1], and corrected with the statement that, in fact, the coupling parameter must be sufficiently small, in the style of [La.2]. The first result on exact/approximate boundary controllability of mechanical/thermal variables for the same thermo-elastic plate with controls in the free BC, which does not require any smallness hypothesis on the model, is [A-L.1]. Here, however, the coefficients, in particular $\alpha$, are constant. This assumption is also critical to the arguments of [A-L.1], as it allows for the introduction of a certain transformation of variables, to make the problem more tractable. On the other hand, it is known that observability/controllability estimates are sensitive with respect to variable perturbations of the “energy level” terms in the equations. In fact, even for simple plate equations or wave equations, energy methods (multipliers) [Tr.1, L-T.1, Li.1, K.1] used to obtain the right continuous observability estimates are not adequate in the presence of variable coefficients at the energy level (as the one represented by $\alpha$ in (1.1)). More sophisticated methods are called for: [B-L-R.1, L-T.6, Ta.1, Ta.2, L-T-Y.1, F-I.1], etc. Thus, it is expected that similar difficulties will recur in the study of thermo-elasticity. Thus, a main contribution of this paper is the presence of a variable thermal coefficient $\alpha$ without any smallness requirement for the present boundary control case where, moreover, the non-critical controls are arbitrarily smooth.

The above features dictate the necessity of introducing new mathematical tools into the study of this boundary control problem, which include: new uniqueness theorems [E-L-T.2, E-L-T.3] (of interest in their own right) for both elastic Kirchoff equations and thermo-elastic equations, a new functional analytic setting of the problem, new PDE Carlemann-type estimates, in the case of variable $\alpha$, etc.

Altogether different is the problem of exact null-controllability. In the case of a one-dimensional thermoelastic equation with hinged boundary conditions, an exact null controllability result is given in [H-Z.1] by use of the moment problem approach, via a scalar boundary control. For a more general thermoelastic model (in any dimension) this time with distributed control either in the mechanical (Kirchoff) equation, or else in the thermal equation (only), exact-null controllability is obtained in [L-T.12].

Finally, a very recent exact-null controllability paper (of which we have become aware after completion of the first draft of the present work) is [A-T.1], which deals, however, with a thermoelastic wave (rather than plate) system, which couples a wave equation with a thermal equation, with boundary control in the Dirichlet B.C. of the wave equation. It is likely (though the details may be demanding) that the treatment of [A-T.1] could be adapted to prove the exact null controllability of thermoelastic plates.
with \textit{hinged} boundary controls. The technical Carleman estimates of [A-T.1] neither imply, nor are implied by, the results of the present paper. In particular, the problem in [A-T.1] does not need backward uniqueness results, which are instead critical for our problem.

2. THERMO-ELASTIC WELL-POSEDNESS AND DUAL PROBLEM

\textbf{Well-Posedness: Homogeneous Problem}

Let, at first, \( u_1 = u_2 = u_3 = 0 \) in (1.2). We introduce the following operators and spaces (with equivalent norms):

\[ \mathcal{A} f = -\Delta f, \quad \mathcal{A}_y = (I + \gamma \mathcal{A}), \quad \mathcal{D}(\mathcal{A}_y) = \mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H^1_0(\Omega); \quad (2.1a) \]

\[ \mathcal{D}(\mathcal{A}^{1/2}) = \{ f \in H^1(\Omega) : f|_\Gamma = \Delta f|_\Gamma = 0 \}; \quad (2.1b) \]

\[ \mathcal{D}(\mathcal{A}_y^{1/2}) = \mathcal{D}(\mathcal{A}^{1/2}) = H^1_0(\Omega), \quad (2.2) \]

\[ (x_1, x_2)_{\mathcal{D}(\mathcal{A}_y^{1/2})} = ((I + \gamma \mathcal{A})x_1, x_2)_{H}, \quad H = L_2(\Omega); \]

\[ Y_\gamma = Y_{1, \gamma} \times H = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times H \]

\[ \equiv [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega) \times L_2(\Omega). \]

\textbf{Lemma 2.1} [E-L-T.1, E-L-T.2]. Let \( u_1 = u_2 = u_3 = 0 \) in (1.2). Then: (i) Problem (1.1), (1.2) defines an s.c. semigroup denoted by \( e^{\mathcal{A}_y t} \):

\[ y_0 = \{ u_0, w_1, \theta_0 \} \rightarrow e^{\mathcal{A}_y t} y_0 = \{ w(t), w(t), \theta(t) \} \text{ in the space } Y_\gamma \text{ given by (2.3), whose generator is given explicitly in [E-L-T.1, E-L-T.2].} \]

The dual semigroup dynamics \( \bar{y}_0 = \{ \phi_0, -\phi_1, \eta_0 \} \rightarrow e^{\mathcal{A}^{1/2} t} \bar{y}_0 = \{ \phi(t), -\phi_y(t), \eta(t) \} \text{ whose generator is given explicitly in [E-L-T.1, E-L-T.2]} \text{ is given by the following thermo-elastic (dual) problem:} \]

\[ \phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi - \text{div}(\alpha(x)\nabla \eta) \equiv 0 \quad \text{in } Q, \quad (2.4a) \]

\[ \eta_t - \Delta \eta + \text{div}(\alpha(x)\nabla \phi_t) \equiv 0 \quad \text{in } Q, \quad (2.4b) \]

\[ \phi(0, \cdot) = \phi_0; \quad \phi_t(0, \cdot) = \phi_1, \quad \eta(0, \cdot) = \eta_0 \quad \text{in } \Omega, \quad (2.4c) \]

\[ \phi = \Delta \phi \equiv 0, \quad \eta = 0 \quad \text{on } \Sigma. \quad (2.4d) \]

\textbf{Well-Posedness: Non-homogeneous Problem}

The following result is important for the present paper in (4.0.1b) below [Tr.2, Theorems 1.1]. See also [H-Z.1] in the one-dimensional hinged case.
458 ELLER, LASIECKA, AND TRIGGIANI

 Proposition 2.2. With reference to problem (1.1), (1.2), let $u_1 \equiv u_5 \equiv 0$, $w_0 = w_1 = \theta_0 = 0$. Then the following maps are continuous:

$$u_2 \in L_2(0, T; L_2(\Gamma))$$

\[ \Rightarrow \{w, w_t, \theta\} \in C\left([0, T]; \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2})\right) \]

\[ = C\left([0, T]; [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H_0^1(\Omega)\right), \quad (2.5a) \]

\[ w_{tt} \in L_2(0, T; L_2(\Omega)), \]

\[ \theta_t \in L_2(0, T; L_2(\Omega)) \cap C\left([0, T]; [\mathcal{D}(\mathcal{A}^{1/2})]\right). \quad (2.5b) \]

The controllability result of Theorem 1.1 is consistent with the regularity results of Proposition 2.2.

3. ASSOCIATED KIRCHHOFF-EQUATION: STRUCTURAL DECOMPOSITION OF THE s.c. SEMIGROUP $e^{\mathcal{A}_t}$, $\gamma > 0$

Associated Kirchoff Equation

When $\gamma > 0$, the thermo-elastic plate has a hyperbolic-dominated character, in the sense of the next result. Write $\text{div}(\alpha(x)\nabla \theta) = \alpha(x)\Delta \theta + \nabla \alpha \cdot \nabla \theta$ in the first equation (1.1a), and substitute $\Delta \theta$ from the second Eq. (1.1b) to obtain

$$w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w - \alpha \text{div}(\alpha \nabla w_t) = -\alpha \theta_t - \nabla \alpha \cdot \nabla \theta \quad \text{in } Q. \quad (3.1)$$

This, then, induces one to introduce the purely mechanical Kirchoff equation

$$v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v = \alpha \text{div}(\alpha \nabla v_t) \equiv 0 \text{ in } Q; \quad (3.2a)$$

$$v|_{\Sigma} = u_1, \quad \Delta v|_{\Sigma} = u_2. \quad (3.2b)$$

For use in the analysis below, we introduce the following operator $F_a$:

$$F_a f \equiv \alpha \text{div}(\alpha \nabla f), \quad \mathcal{D}(F_a) = H^2(\Omega) \cap H_0^1(\Omega). \quad (3.3)$$

Its adjoint $F_a^*$, in the sense that $(F_a f, g)_{L_2(\Omega)} = (f, F_a^* g)_{L_2(\Omega)}$, $\forall f \in \mathcal{D}(F_a)$, $g \in \mathcal{D}(F_a^*)$, is given by

$$F_a^* g = \text{div}(\alpha \nabla (\alpha g)), \quad \mathcal{D}(F_a^*) = \mathcal{D}(F_a) = H^2(\Omega) \cap H_0^1(\Omega). \quad (3.4)$$

To show (3.4), one starts with (3.3) and applies the divergence theorem twice [E-L-T2].
Homogeneous Case \( u_1 = u_2 = 0 \) in (3.2b)

The abstract version of problem (3.2) with \( u_1 = u_2 = 0 \) in (3.2b) (hinged BC) is given, via (2.1a), (3.3), by

\[
v_t + \gamma \Delta v + \delta^2 v - F_v v_t = 0,
\]

with more details given in [E-L-T.1, E-L-T.2].

**Lemma 3.1.**

(i) Problem (3.2) generates an s.c. group \( e^{\mathcal{A}_1, t} \) on \( Y_{1, \gamma} \) (see (2.3)): \( \tilde{v}_0 = \{ v_0, v_1 \} \Rightarrow e^{\mathcal{A}_1, t} \tilde{v}_0 = \{ v(t), v_1(t) \} \), where \( v \) solves (3.2) with \( u_1 = u_2 = 0 \). Its generator \( \mathcal{A}_{1, \gamma} \) is explicitly given in [E-L-T.1, E-L-T.2].

(ii) The adjoint s.c. group \( e^{\mathcal{A}_1, t} \) whose generator \( \mathcal{A}_{1, \gamma}^* \) is explicitly given in [E-L-T.1, E-L-T.2], describes the following dynamics: \( \psi_0 = \{ \psi_0, -\psi_1 \} \rightarrow e^{\mathcal{A}_1, t} \psi_0 = \{ \psi(t), -\psi_1(t) \} \), where \( \psi \) solves the (dual) Kirchoff problem

\[
\psi_t - \gamma \Delta \psi_t + \Delta^2 \psi - \text{div}(\alpha \nabla(\alpha \psi_t)) = 0,
\]

\[
\psi = \Delta \psi \equiv 0 \text{ on } \Sigma,
\]

whose abstract version is, via (2.1), (3.4),

\[
\psi_{tt} + \gamma \Delta \psi_{tt} + \Delta^2 \psi - F^*_\alpha \psi_t = 0 \quad \text{(hinged BC)}.
\]

**Structural Decomposition of** \( e^{\mathcal{A}_1, t} \)

The following result, critical for our present development, holds true.

**Theorem 3.2.** Consider the thermo-elastic semigroup \( e^{\mathcal{A}_1, t} \) on \( Y_{1, \gamma} \) of Lemma 2.1 and the associated Kirchoff group \( e^{\mathcal{A}_{1, \gamma}, t} \) on \( Y_{1, \gamma} \) of Lemma 3.1. The following structural decomposition holds true for any \( t > 0 \), and \( \{w_0, w_1, \theta_0\} \in Y_{\gamma}, \)

\[
\begin{bmatrix}
w(t) \\
w_1(t) \\
\theta(t)
\end{bmatrix} = e^{\mathcal{A}_1, t} \begin{bmatrix}
w_0 \\
w_1 \\
\theta_0
\end{bmatrix} = \begin{bmatrix}
e^{\mathcal{A}_1, t} \begin{bmatrix}
w_0 \\
w_1 \\
0
\end{bmatrix} + \mathcal{K}_{\gamma}(t) \begin{bmatrix}
w_0 \\
w_1 \\
\theta_0
\end{bmatrix}.
\end{bmatrix}
\]

where \( \mathcal{K}_{\gamma}(t) \) is a compact operator on \( Y_{\gamma} \) for each \( t > 0 \).

(b) Let \( \Pi_m \) be the projection \( Y_{\gamma} \rightarrow Y_{1, \gamma} : [v_1, v_2, v_3] \rightarrow [v_1, v_2] \) onto the mechanical space and let \( \Pi_m^* : [v_1, v_2] \rightarrow [v_1, v_2, 0] \) be its adjoint \( Y_{1, \gamma} \rightarrow Y_{\gamma}. \) Then, with reference to problem (1.1), (1.2), we may write using (3.8)

\[
\begin{bmatrix}
w(T) \\
w_1(T)
\end{bmatrix} = \Pi_m \int_0^T e^{\mathcal{A}_1, (T-t)} \mathcal{B}_h u(t) dt
\]

\[
= \int_0^T e^{\mathcal{A}_1, (T-t)} \Pi_m^* \mathcal{B}_h u(t) dt + Q,
\]

\[
Q \equiv \Pi_m \int_0^T \mathcal{K}_{\gamma}(T-t) \mathcal{B}_h u(t) dt:
\]

compact \( \mathcal{U} \rightarrow X_{1, \gamma} = Y_{1, \gamma} = [H_2^2(\Omega) \times H^1_0(\Omega)] \times H^1_0(\Omega). \)
Above, $\mathcal{B}_h$ is the control operator associated with the boundary control problem (1.1), (1.2), which is explicitly identified in [E-L-T2, Appendix B]. Part (a) of the above is a special case of a more general structural decomposition result given in [L-T7, Theorems 1.2.1, 1.2.2]. See also [L-T8, Appendix B]. As a matter of fact, [L-T7] considered explicitly the case of constant $\alpha$. A variable $\alpha$ in space produces the additional contribution $\nabla\alpha \cdot \nabla \theta$ in (3.1) (not present in [L-T7]), which, however, still yields a compact additional contribution to the operator denoted by $L_t$ in [L-T7, Eq. (3.5)]. Thus, [L-T7, Theorem 1.2.2] applies to the present case as well yields Part (a) of Theorem 3.2.

However, it is Part (b) of Theorem 3.2 that will be critically used in this paper (see Section 4). Compactness of $Q$ in the present hinged case is given in [L-T7, Proposition 6.2.1, p. 62] with a sketch of the proof. A more detailed expansion of this proof is given below. We note, preliminarily, that compactness of $X_\gamma(t)$ does not suffice to claim compactness of $Q$, as stated in (3.10b), since the control operator $\mathcal{B}_h$ is (highly) unbounded (if $\mathcal{B}_h$ were bounded, as in the distributed control case [T-Z1], compactness of $Q$ would follow by Mazur’s theorem as in [L-T7]).

**Proof of Theorem 3.2, Part (b): Compactness of $Q$** (after [L-T7, Proposition 6.2.1, p. 62]). We may set $u_1 \equiv u_3 \equiv 0$ since these controls belong to a $C_\infty$-class. As we concentrate only on $u_2 \in L_2(\Sigma)$, we may set $\mathcal{B}_h = \Pi^*_m \mathcal{B}_m$, where $\mathcal{B}_m$ denotes the first two components of $\mathcal{B}_h$, while the third component of $\mathcal{B}_h$ is zero [E-L-T2, Appendix B]. In this case, instead of (3.10b), we show equivalently that

$$Q^* = \mathcal{B}^*_m \Pi M^* X_\gamma^*(\cdot) \Pi^*_m; \text{ compact } X_{1, \gamma} \equiv Y_{1, \gamma} \to L_2(0, T; L_2(\Gamma)). \quad (3.11)$$

(The notation in (3.11) agrees with that of [L-T7, Eq. 6.2.9]), except that in this latter reference we chose—in line with the emphasis of that paper—to start with the dual/adjoint thermo-elastic semigroup, rather than the original one of the present paper.) We now prove (3.11).

**Step 1.** Taking the $Y_\gamma$-adjoint of (3.8), we see from [L-T7, Eqs. (3.35) and (3.5), expressed now for the adjoint problem, that if $\tilde{y}_0 = [w_0, w_1]$, $\Pi^*_m \tilde{y}_0 = y_0 = [w_0, w_1, 0]$, we have, explicitly

$$\mathcal{B}_m^* \Pi M^* \mathcal{B}_m^*(t) \Pi^*_m \tilde{y}_0 = \mathcal{B}_m^* \int_0^t e^{A_\gamma^*(\cdot-t)}$$

$$\begin{bmatrix}
0 \\
-\mathcal{B}_m^* [\eta_1(\tau; y_0) + \nabla \alpha \cdot \nabla \eta(\tau; y_0)]
\end{bmatrix} d\tau,$$

(3.12)

where $\eta$ is the thermal component of the adjoint thermo-elastic problem (2.4).
We next use, critically, the following trace regularity for the Kirchoff homogeneous problem (3.6):

$$\mathcal{R}_m e^{\mathcal{A}_i \gamma t}: \text{continuous } Y_{1, \gamma} \to L_2(0, T; L_2(\Gamma)).$$

(3.13)

By duality, (3.13) is equivalent to the following interior regularity of the corresponding Kirchoff non-homogeneous problem (3.2),

$$v_0 = v_1 = 0, \quad u_1 \equiv 0, \quad u_2 \in L_2(\Sigma) \to \{v, v_t\} \in C([0, T]; Y_{1, \gamma})$$

(3.14)

(see (2.3)), which is true by [L-T.4]. (The proof for \(\alpha \equiv 1\) of this latter reference extends verbatim to the case of \(\alpha\) variable and smooth.) Thus, returning to (3.11), (3.12), we estimate by virtue of (3.13) with \(U = L_2(\Gamma)\), via Schwarz inequality and a change in the order of integration,

$$\|Q^* \hat{y}_0\|_{L_2(\Sigma)}^2 = \int_0^T \left\| \mathcal{R}_m e^{\mathcal{A}_i \gamma t} \right\|^2_U dt \leq T \int_0^T \int_0^t \left\| -\mathcal{A}_i^{-1}[\eta_t(\tau; y_0) + \nabla \alpha \cdot \nabla \eta(\tau; y_0)] \right\|^2_U d\tau dt$$

(3.15)

$$= T \int_0^T \int_{\tau}^T \left\| -\mathcal{A}_i^{-1}[\eta_t(\tau; y_0) + \nabla \alpha \cdot \nabla \eta(\tau; y_0)] \right\|^2_U d\tau d\tau$$

(3.16)

(by (3.13))

$$\leq C_T T \int_0^T \left\| -\mathcal{A}_i^{-1}[\eta_t(\tau; y_0) + \nabla \alpha \cdot \nabla \eta(\tau; y_0)] \right\|^2_{Y_{1, \gamma}} d\tau$$

(3.17)

(by (2.2))

The key trace estimate (3.13) has been used in going from (3.18) to (3.19). (Equation (3.19) coincides with [L-T.7, Eq. (6.2.10)] for which only a sketch was given, now elaborated in (3.15) through (3.19).)

**Step 2.** The map

$$y_0 = [w_0, w_1, 0] \in Y_\gamma \to \mathcal{A}_i^{-1/2}[\eta_t + \nabla \alpha \cdot \nabla \eta] \in L_2(0, T; L_2(\Omega))$$

(3.20)

is compact.
This step can be proved as in [L-T, p. 62] by Aubin’s lemma [A.1], due to the key regularity properties $\eta_t, |\nabla \eta| \in L_2(0, T; L_2(\Omega))$ [L-T; E-L-T, Sect. 9; E-L-T].

Step 3. Using (3.19) in (3.18) yields (3.11), as desired.

Abstract Model of Non-homogeneous Problem (3.2)

This is given explicitly in [E-L-T].

4. CONSEQUENCE OF THE STRUCTURAL DECOMPOSITION: A STRATEGY FOR THE CONTROLLABILITY PROBLEM

4.0. Preliminaries

[L-T Sect. 6] presents a strategy, essentially already used in [L-T, pp. 119–120], to obtain an exact controllability (surjectivity) result. In the present case of thermo-elastic plates, its applicability is based on the structural decomposition Theorem 3.2, combined with a soft argument as in [E-L-T Appendix D, A-L.1]. This is amply elaborated in [L-T, Sect. 6.2]. With reference to the thermo-elastic plate (1.1), we take henceforth zero initial condition $y_0 = \begin{bmatrix} w_0 \end{bmatrix}$ and boundary controls $u = \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix}$ in (1.2), of the same class $U$ as specified in Theorem 1.1: i.e., $u_1 \in C_0^\infty(\Sigma_1)$, $u_3 \in C_0^\infty(\Sigma_3)$, and $u_2 \in L_2(\Sigma_2).$ We then define the input-solution operator $\mathcal{L}_T$ at the terminal time $T$ by

$$\mathcal{L}_T u = \{ w(T), w(t), \theta(T) \} = \int_0^T e^{A(T-t)}Bu(t) dt :$$ (4.0.1a)

continuous $\mathcal{U} \to X_\gamma \equiv Y_{1, \gamma} \times H_0^1(\Omega),$ (4.0.1b)

$$X_\gamma \equiv Y_{1, \gamma} \times H_0^1(\Omega) \equiv [H^2(\Omega) \cap H_0^1(\Omega)]$$

$$\times H_0^1(\Omega) \times H_0^1(\Omega) \subset Y_\gamma$$ (4.0.1c)

(see (2.3)), where the asserted regularity in (4.0.1b) follows (mostly) from Proposition 2.2, where $u_1 \equiv u_3 \equiv 0,$ $u_2 \in L_2(\Sigma).$ The precise form of the boundary $\to$ interior operator $\mathcal{B}$ is given in [E-L-T, Appendix B]. Let $\Pi_m$ be the projection (see (2.3)) $Y_\gamma \to Y_{1, \gamma}; \begin{bmatrix} v_1, v_2, v_3 \end{bmatrix} \to \begin{bmatrix} v_1, v_2 \end{bmatrix}$ onto the mechanical state space and let $\Pi_m^*: \begin{bmatrix} v_1, v_2 \end{bmatrix} \to \begin{bmatrix} v_1, v_2, 0 \end{bmatrix}$ be its adjoint $Y_{1, \gamma} \to Y_\gamma.$ The strategy for controllability, as stated in Theorem 1.1, hinges on the following two steps.

Step 1. Show exact controllability on the space $Y_{1, \gamma}$ in (4.0.1b,c) or (2.3) from the origin at time $t = T$ of the thermo-elastic plate problem (1.1), (1.2) in the mechanical variables; in symbols, with reference to (4.0.1), show that

$$\Pi_m \mathcal{F}_T : \text{surjective } \mathcal{U} \text{ onto } Y_{1, \gamma},$$ (4.0.2)
where \( \mathcal{U} \) is the preassigned space of controls \( u = [u_1, u_2, u_3] \), specified by Theorem 1.1.

**Step 2.** Show approximate controllability on the space \( X_\gamma \) in (4.0.1b,c) from the origin at time \( t = T \) of the thermo-elastic plate (1.1), (1.2); in symbols, show that the range of \( \mathcal{L}_T \) is dense in \( X_\gamma = Y_{1, \gamma} \times Y_2; Y_2 = H^1_0(\Omega) \):

\[
\overline{\mathcal{L}_T \mathcal{U}} \equiv \overline{\mathcal{R}(\mathcal{L}_T)} = X_\gamma, \quad \mathcal{R} = \text{range}. \tag{4.0.3}
\]

Once Steps 1 and 2 are accomplished, a soft argument as in [E-L-T.2, Appendix D, Theorem D.1, A-L.1], where \( \mathcal{L}_T \) is continuous \( \mathcal{U} \to X_\gamma \), as noted in (4.0.1a), then shows the following

**Desired Conclusion**

Steps 1 and 2 imply exact controllability on the space \( Y_{1, \gamma} \) from the origin at time \( t = T \) of the thermo-elastic plate (1.1), (1.2) in the mechanical variable and, simultaneously, approximate controllability on the space \( H^1_0(\Omega) \) from the origin at time \( t = T \) in the thermal variable, i.e., precisely, the statement of Theorem 1.1.

4.1. Implementation of Step 1: Exact Controllability of the Thermo-Elastic Plate Problem in the Mechanical Variables

As explained in [L-T.7, Sect. 6], it is at the level of implementing Step 1 that the structural decomposition of the thermo-elastic semigroup as in Theorem 3.2 is critically used. The key is the following simple result, essentially already used in [L-T.3, pp. 119–120], from approximate to exact controllability.

**Proposition 4.1.1** [L-T.7, Proposition 6.1.1]. Let \( J = S + Q \), and let \( X \) be a Hilbert space, where:

(i) \( J \) is a closed operator \( \mathcal{U} \subset \mathcal{D}(J) \to X \) with dense range \( \overline{\mathcal{R}(J)} = X \) (approximate controllability), equivalently, with trivial null space of the adjoint \( J^* : \mathcal{N}(J^*) = \{0\} \);

(ii) \( S \) is a closed, surjective operator: \( \mathcal{U} \subset \mathcal{D}(S) \) onto \( X \), where \( \mathcal{D}(S) = \mathcal{D}(J) \);

(iii) \( Q \) is a compact operator: \( \mathcal{U} \to X \).

Then, \( J \) is surjective \( \mathcal{U} \subset \mathcal{D}(J) \) onto \( X \) (exact controllability).

To implement Step 1 to our problem, and with reference to the decomposition (3.8) of Theorem 3.2, we return to (4.1) and take in
agreement with Theorem 3.2(b), Eq. (3.9), (3.10):

\( J_t \equiv \Pi_m (\mathcal{L}_T u) \equiv \Pi_m \int_0^T e^{\mathcal{A}_1 (T-t)} \mathcal{B} u(t) dt, \) \hspace{1cm} (4.1.1)

\( S_t \equiv \int_0^T e^{\mathcal{A}_1 (T-t)} \Pi_m \mathcal{B} u(t) dt \overset{\text{def}}{=} \mathcal{L}_m T u, \) \hspace{1cm} (4.1.2)

\( Q_t \equiv \Pi_m \int_0^T \mathcal{K}(T-t) \mathcal{B} u(t) dt. \) \hspace{1cm} (4.1.3)

Assumption (i) of Proposition 4.1.1 then means that \( \Pi_m \mathcal{L}_T \) has dense range in \( Y_1, \) but this is, \( a \text{ fortiori}, \) assured by the more demanding condition of Step 2.

Assumption (iii) of Proposition 4.1.1 on compactness of the operator \( Q \) in (4.1.3) is asserted in (3.10b).

Finally, one needs to verify Assumption (ii) of Proposition 4.1.1 on the operator \( S \) in (4.1.2). More precisely, to this end, one needs to establish the following exact controllability results of the Kirchoff equation (3.2).

**Theorem 4.1.2.** Let \( \emptyset \neq \Gamma_1, \Gamma_2 \subset \Gamma \) be open subsets of the boundary \( \Gamma, \) with non-empty intersection of positive measure (we think of \( \Gamma_1 \) as arbitrarily small, so that, without loss of generality, we may always assume that \( \emptyset \neq \Gamma_1 \subset \Gamma_2 \)). Moreover, regarding \( \Gamma_2, \) we assume that: there exists a point \( x_0 \in \mathbb{R}^2 \), such that

\[(x - x_0) \cdot \nu(x) \leq 0 \quad \text{for} \quad x \in \Gamma \setminus \Gamma_2, \] \hspace{1cm} (4.1.4)

with \( \nu(x) \) the unit outward normal at \( x \in \Gamma \). Let

\[ T_{0, h} = 2\sqrt{\gamma} \max_i \sup_{x \in \Omega} \text{dist}(x, \Gamma_i), \quad i = 1, 2 \] \hspace{1cm} (4.1.5)

(\( h \) stands for “hinged”). Let \( \alpha \in C^2(\overline{\Omega}) \). Finally, let \( \{v_0, v_1\} \) and \( \{v_{0, T}, v_{1, T}\} \) be pre-assigned initial and target states of the (mechanical) \( v \)-problem (3.2), with

\[
\{v_0, v_1\} \quad \text{and} \quad \{v_{0, T}, v_{1, T}\} \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega). \] \hspace{1cm} (4.1.6)

Then, for any \( T > T_{0, h} \), there exist control functions

\[
u_1 = \begin{cases} \bar{u}_1 \in C^\infty_0(\Sigma_1) & \text{on} \quad \Sigma - \Sigma_1, \\ 0 & \text{on} \quad \Sigma - \Sigma_2 \end{cases}, \quad u_2 = \begin{cases} \bar{u}_2 \in L^2(\Sigma_2) & \text{on} \quad \Sigma - \Sigma_1, \\ 0 & \text{on} \quad \Sigma - \Sigma_2 \end{cases}, \] \hspace{1cm} (4.1.7)

such that the solution corresponding to Eq. (3.2a) with controls \( \{u_1, u_2\} \) in (4.1.7) in the hinged BC (3.2b) (left) satisfies the terminal condition

\[ v(T) = v_{0, T}, \quad v_i(T) = v_{1, T}. \] \hspace{1cm} (4.1.8)
The proof of Theorem 4.1.2 is given in two steps as announced in [E-L-T1, Sect. 5].

First, in Section 5, the property of exact controllability for the \( v \)-problem (3.2) claimed in Theorem 4.1.2 is reformulated, by duality, as an equivalent continuous observability inequality (Eq. (5.11)), for the dual homogeneous \( \psi \)-problem (3.6). Next, Section 6 establishes validity of the continuous observability inequality (5.11) for \( \psi \); see Theorem 6.3.

Remark 4.1.1. We note that Theorem 4.1.2 does not follow from known results; see [La.1, L-T.4, L-L.1, Li.1, K.1], and references therein. Indeed, the two main novelties of Theorem 4.1.2 over known literature are: (i) the coefficient \( \alpha \) is space variable dependent and, consequently, (ii) the control function \( u_1 \) has arbitrarily small support on \( \Gamma \). These two factors contribute additional technical difficulties and the techniques/methods in the quoted literature are no longer directly applicable.

Once Theorem 4.1.2 is established—in Section 6—application of the abstract Proposition 4.1.1 specialized to (4.1.1)–(4.1.3) yields the desired exact controllability of the thermo-elastic plate (1.1), (1.2) in the mechanical variables only in the space \( Y_{1,\gamma} = \Pi_m Y_\gamma \) of the mechanical variables \( \{w, w_1\} \), by means of the boundary controls specified in (1.6).

Theorem 4.1.3. Let \( T > T_{0,h} \); see (4.1.5). Under the assumptions and setting of Theorem 4.1.2, we have that the thermo-elastic plate problem (1.1), (1.2) is exactly controllable on the space \( Y_{1,\gamma} = \Pi_m Y_\gamma \) of the mechanical variables \( \{w, w_1\} \), by means of the boundary controls specified in (1.6).

We next provide the continuous observability inequality, for the dual thermo-elastic problem (2.4), which corresponds to the property of exact controllability (a fortiori established in Theorem 4.1.3) of the thermo-elastic problem (1.1), (1.2), with boundary controls \( \{u_1, u_2, u_3\} \) of a clean, larger class than that specified in (1.6).

Corollary 4.1.4. Let \( T > T_{0,h} \); see (4.1.5). Then, according to Theorem 4.1.3, the thermo-elastic mixed problem (1.1), (1.2) (hinged case) is exactly controllable on \([0, T]\) in the mechanical variables \( \{w, w_1\} \), in the state space \( Y_{1,\gamma} = \Pi_m Y_\gamma,h = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \) (see (2.3)), within the class \( \mathcal{U} \) of controls (compare with (1.6) and with [E-L-T.2, Appendix C, Eq. (C.2b)]

\[
\begin{align*}
        u_1 &\in H_0^2(0, T; L_2(\Gamma_1)), & u_2 &\in L_2(0, T; L_2(\Gamma_2)), \\
    u_3 &\in L_2(0, T; L_2(\Gamma_3)),
\end{align*}
\]

(4.1.9)

with \( \emptyset \neq \tilde{\Gamma} \equiv \Gamma_1 = \Gamma_3 \subset \Gamma_2 \), as in the last statement of Remark 1.1. Equivalently, the following continuous observability inequality holds true for...
the dual thermo-elastic problem (2.4), with initial condition \( \{ \phi_0, \phi_1 \} \in Y_{1, \gamma} \) and \( \eta_0 = 0 \) at \( t = T \): there exists a constant \( C_T > 0 \) such that

\[
\left\| \frac{\partial \phi_t}{\partial v} \right\|_{L^2(0,T;L^2(\Gamma_t))}^2 + \left\| \frac{\partial \eta}{\partial v} \right\|_{L^2(0,T;L^2(\Gamma_t))}^2 + \left\| \frac{\partial \Delta \phi_t}{\partial v} \right\|_{[H^1_0(0,T;L^2(\Gamma_t))]}^2 \\
\geq C_T \| \{ \phi_0, \phi_1 \} \|_{Y_{1, \gamma}}^2,
\]

(4.1.10)

(duality with respect to \( L^2(0,T;L^2(\Gamma_t)) \)).

Proof.

Step 1. The surjectivity condition (4.0.2) (a restatement of the exact controllability property of the present corollary), with class \( \mathcal{U} \) as in (4.1.9), is equivalent, by a standard result [T-L.1, p. 235], to the following inequality: there exists \( C_T > 0 \) such that

\[
\left\| \mathcal{L}_T^* \Pi_n^* \left[ \begin{array}{c} \phi_0 \\ \phi_1 \end{array} \right] \right\|_{Y_{1, \gamma}} \geq C_T \| \{ \phi_0, \phi_1 \} \|_{Y_{1, \gamma}},
\]

(4.1.11)

where \( \Pi_n^*[\phi_0, \phi_1] = [\phi_0, \phi_1, 0] \); see below (4.0.1).

Step 2. By [E-L-T.2, identity (C.5) of Lemma C.1 in Appendix C] inequality (4.1.11) is equivalent to: there exists a constant \( C_T > 0 \) such that the solution of the dual thermo-elastic problem (2.4) with initial condition \( \{ \phi_0, \phi_1 \} \in Y_{1, \gamma} \) and \( \eta_0 = 0 \) at time \( t = T \) satisfies for \( T > T_{0, \kappa} \) and \( \Sigma_i = (0, T] \times \Gamma_i, i = 2, 3 \):

\[
\left\| \frac{\partial \phi_t}{\partial v} \right\|_{L^2(\Sigma_2)}^2 + \left\| \frac{\partial \Delta \phi_t}{\partial v} + \alpha \frac{\partial \eta_t}{\partial v} - \gamma \frac{\partial \phi_{tt}}{\partial v} \right\|_{[H^1_0(0,T;L^2(\Gamma_t))]}^2 \\
+ \left\| \frac{\alpha \partial \phi_t}{\partial v} - \frac{\partial \eta_t}{\partial v} \right\|_{L^2(\Sigma_1)}^2 \geq C_T \| \{ \phi_0, \phi_1 \} \|_{Y_{1, \gamma}}^2.
\]

(4.1.12)

Step 3. Inequality (4.1.12) is, in turn, equivalent to the claimed inequality (4.1.10). This is so since with \( \emptyset \neq \tilde{\Gamma} = \Gamma_1 = \Gamma_3 \subseteq \Gamma_2 \), and thus \( \tilde{\Sigma} = \Sigma_1 = \Sigma_3 \subseteq \Sigma_2 \), as assumed, we have

\[
\left\| \frac{\partial \Delta \phi_t}{\partial v} + \alpha \frac{\partial \eta_t}{\partial v} - \gamma \frac{\partial \phi_{tt}}{\partial v} \right\|_{[H^1_0(0,T;L^2(\Gamma_t))]}^2 \\
+ \left\| \alpha \frac{\partial \phi_t}{\partial v} - \frac{\partial \eta_t}{\partial v} \right\|_{L^2(\Sigma_1)}^2 \leq C_{\alpha, \gamma} \left\{ \left\| \frac{\partial \phi_t}{\partial v} \right\|_{H^1_0(0,T;L^2(\Gamma_t))}^2 + \left\| \frac{\partial \eta_t}{\partial v} \right\|_{L^2(\Sigma_1)}^2 + \left\| \frac{\partial \phi_t}{\partial v} \right\|_{L^2(\Sigma_2)}^2 \right\}
\]

(4.1.13)
in one direction (4.1.12) ⇒ (4.1.10). As to the other direction, we first note that
\[
\left\| \frac{\partial \eta}{\partial \nu} \right\|_{L_2(\tilde{\Sigma})} \leq \left\| \alpha \frac{\partial \phi_t}{\partial \nu} - \frac{\partial \eta}{\partial \nu} \right\|_{L_2(\tilde{\Sigma})} + C_a \left\| \frac{\partial \phi_t}{\partial \nu} \right\|_{L_2(\tilde{\Sigma})}, \quad (4.1.14)
\]
and then (4.1.10) implies (4.1.12), as desired. Corollary 4.1.4 is proved.

4.2. Implementation of Step 2 by Duality: Unique Continuation of Over-determined Dual Thermo-Elastic Systems

Our task in this section is to address the denseness condition (4.0.3), \( \mathcal{R}(\mathcal{L}_T) = X_\gamma \) of Step 2, and recast it in an equivalent, more amenable form, by duality.

**Dual Version: Uniqueness Property of an Over-determined Plate Problem**

By duality, via the input-solution operator \( \mathcal{L}_T \) in (4.0.1), the denseness condition (4.0.3) is equivalent the injectivity (observability) condition
\[
\mathcal{N}(\mathcal{L}_T^*) = \{0\} \quad \text{or} \quad \{\mathcal{L}_T^* \tilde{y}_0\}(t) \equiv \mathcal{B}^* e^{\mathbb{L}_T^*(T-t)} \tilde{y}_0 \equiv 0, \quad 0 \leq t \leq T, \quad \tilde{y}_0 \in X_\gamma
\]
\[
\Rightarrow \tilde{y}_0 = 0, \quad (4.2.1)
\]
where \( \mathcal{N} \) denotes the null space, and the space \( X_\gamma \) is defined in (4.0.1c).
We have seen in Section 2 that \( e^{\mathcal{A}_0 t} \tilde{y}_0 \) is the abstract version of the thermoelastic dual problem (2.4) in \( \{ \psi, \eta \} \) with hinged BC, as in (2.4d) over all of \( \Sigma \). A precise version of the abstract injectivity condition (4.2.1) in terms of the traces of the solution \( \{ \phi, \eta \} \) of problem (2.4), with the respective BC (2.4d), is given in [E-L-T 2, Appendix C, Lemma C.1]. It provides the following result:

**Proposition 4.2.1.** Let the operator \( \mathcal{L}_T \) in (4.0.1) be defined as

\[
\mathcal{L}_T : H_0^2(0, T; L_2(\Gamma)) \times L_2(\Sigma) \times L_2(\Sigma) \ni u \rightarrow X_\gamma \subset Y_\gamma,
\]

with \( Y_\gamma \) given by (2.3). Then condition (4.2.1) is equivalently restated as follows: Let \( \{ \phi, \eta \} \in C([0, T]; X_\gamma) \) be a solution of the following overdetermined problem: with \( \emptyset \neq \tilde{\Gamma} \equiv \Gamma_1 = \Gamma_3 \subset \Gamma_2 \) (Remark 1.1),

\[
\begin{align*}
\phi_{tt} - \gamma \Delta \phi_{tt} + \Delta^2 \phi - \text{div}(\alpha(x)\nabla \eta) & = 0 \quad \text{in } Q; \\ \eta_t - \Delta \eta + \text{div}(\alpha(x)\nabla \phi_t) & = 0 \quad \text{in } Q; \\ |\phi|_\Sigma & = 0, \quad |\Delta \phi|_\Sigma = 0, \quad |\eta|_\Sigma = 0 \quad \text{on } \Sigma; \quad \frac{\partial \Delta \phi_t}{\partial \nu} |_{\Sigma_1} & = 0 \quad \text{on } \Sigma_1; \\ \frac{\partial \phi_t}{\partial \nu} |_{\Sigma_2} & = 0 \quad \text{on } \Sigma_2; \quad \frac{\partial \eta}{\partial \nu} |_{\Sigma_3} & = 0 \quad \text{on } \Sigma_3.
\end{align*}
\]

Then the initial condition vanishes:

\[
\{ \phi_0, \phi_1, \eta_0 \} = 0 \quad \text{and hence } \phi \equiv 0, \quad \eta \equiv 0 \quad \text{in } Q.
\]

**Proof.** See [E-L-T 2, Lemma C.1, Appendix C].

With reference to the overdetermined problem (4.2.3), the following uniqueness result, Theorem 4.2.2, holds true for \( T \) sufficiently large.

**Remark 4.2.1.** To put Theorem 4.2.2 below into perspective, we remark that the only uniqueness result for thermoelastic plates available in the literature is the recent one due to Isakov [I.1]: this requires, however, zero Cauchy data on all of \( \Sigma = (0, T] \times \Gamma \), and concludes with the statement that, for \( T > 0 \) sufficiently large, a solution \( \{ \phi, \eta \} \in H^2(Q) \times H^2(Q) \) of problem (4.2.3a-c) satisfies

\[
\phi(T/2, \cdot) = \phi_t(T/2, \cdot) = \eta(T/2, \cdot) = 0.
\]

This is not enough for our present controllability purposes, where, moreover, one needs to conclude that, in fact,

\[
\{ \phi_0, \phi_1, \eta_0 \} = 0, \quad \text{and hence } \phi \equiv 0, \quad \eta \equiv 0 \quad \text{in } Q
\]

(actually in \( (0, \infty) \times \Omega \)).
Fortunately, the passage from (4.2.5) to (4.2.6) holds true. It is a fortiori implied by a recent backward uniqueness theorem [L-R-T] which applies to general thermo-elastic plate equations with space variable coefficients (even in the principal parts) to include problem (4.2.3a,b), and under all canonical BC, in particular, hinged and clamped BC. Paper [L-R-T] was precisely motivated by the result (4.2.5) in [I.1]. For hinged B.C. and constant \( \alpha \), the required backward uniqueness would follow, instead, from [C-T .1].

**Theorem 4.2.2** (unique continuation for (4.2.3)). Assume that \( \{ \phi, \eta \} \in H^4(Q) \times H^2(Q) \) is a solution to the over-determined problem (4.2.3), with zero Cauchy data in \( \Sigma = (0, T] \times \Gamma, \emptyset \neq \tilde{\Gamma} \subset \Gamma = \partial \Omega, \tilde{\Gamma} \) being open and of positive measure, as in (4.2.3c). (We could alternatively start with a pair \( \{ \phi, \eta \} \in H^3(Q) \times H^2(Q) \) of lower regularity which solves (4.2.3a,b), as well as (4.2.3c) with the time derivative sign omitted.) Let \( T > \tilde{T} \), where \( \tilde{T} \) is defined by

\[
\tilde{T} = 2\sqrt{\gamma} \sup_{x \in \Gamma} \text{dist}(x, \tilde{\Gamma}).
\]  

(4.2.7)

Then:

(i) The vanishing at \( t = T/2 \), \( \phi(T/2, \cdot) = \phi(T/2, \cdot) = \eta(T/2, \cdot) = 0 \) as in (4.2.5), holds true.

(ii) Moreover, since problem (4.2.3) with hinged/Dirichlet BC for \( \{ \phi, \eta \} \) on all of \( \Sigma \) (see (4.2.3c)) generates an s.c. semigroup on \( Y_\gamma \) by Lemma 2.1, then the result of [L-R-T] applies, and the unique continuation conclusion (4.2.6) holds true.

The proof of Theorem 4.2.2(i) is given in [E-L-T.2, E-L-T.3].

**Corollary 4.2.3** (unique continuation for (4.2.3)). Let \( \{ \phi, \eta \} \in C([0, T]; Y_\gamma) \) be a solution of either the over-determined problem (4.2.3) in the hinged case, with zero data on \( \tilde{\Sigma} \) as in (4.2.3c), where \( \tilde{\Gamma} = \Gamma_1 = \Gamma_3 \subset \Gamma_2 \) in the hinged case. Assume the geometrical condition (1.3). Then:

(a) we have, in fact, that \( \{ \phi_t, \eta_t \} \in H^3(Q) \times H^1(Q) \);

(b) consequently for \( T > \tilde{T} \) (defined by (4.2.7)), we have that: \( \phi \equiv \eta \equiv 0 \) in \( Q \) as in (4.2.6).

**Proof.** (a) The required boost of regularity in order to apply Theorem 4.2.2 is proved in [E-L-T.2, E-L-T.3].

5. **STEP 1: CONTINUOUS OBSERVABILITY INEQUALITY FOR THE HOMOGENEOUS KIRCHHOFF PROBLEM (3.6)**

In this section we return to the homogeneous Kirchoff problem (3.6) and seek the corresponding continuous observability inequality. By duality, this
inequality is then equivalent to the required exact boundary controllability property of the \(v\)-problem (3.2), with hinged boundary controls, as stated by Theorem 4.1.2.

The Proof of Theorem 4.1.2 Begins

Identifying the continuous observability inequality for problem (3.6) is the first step in the proof of Theorem 4.1.2. To this end, we return to the non-homogeneous problem (3.2) with boundary controls in the hinged BC, whose abstract model is given by [E-L-T2]. Thus, with reference to the map \(L\) explicitly given in [E-L-T2, Eq. (3.23)], we define

\[ L_T u \equiv (Lu)(T) = \int_0^T e^{\partial_0 u, (T-t)\partial_1, h} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} dt, \]  

for \( u = [u_1, u_2] \in H^0_0(\Sigma_1) \times L_2(\Sigma_2) \), where \( \emptyset \not= \Gamma_2 \subset \Gamma_1 \) are the subsets of \( \Gamma \) defined in the assumptions of Theorem 4.1.2. We seek to establish exact controllability (from the origin) of the \(v\)-problem (3.2) at the time \( T \), on the state space \( X_{1,\gamma} = Y_{1,\gamma} = D(\mathcal{A}) \times D(\mathcal{A}_\gamma^{1/2}) \) in (2.3), by means of boundary controls \([u_1, u_2] \in H^0_0(\Sigma_1) \times L_2(\Sigma_2)\), any \( k \geq 2 \), as stated in Theorem 4.1.2. Equivalently, we seek to establish surjectivity of the map \( L_T \):

\[ L_T: H^k_0(\Sigma_1) \times L_2(\Sigma_2) \rightarrow \text{onto} \ X_{1,\gamma} = Y_{1,\gamma} = D(\mathcal{A}) \times D(\mathcal{A}_\gamma^{1/2}). \]  

It is a standard result [T-L.1, p. 235] that the surjectivity condition (5.2) is, in turn, equivalent to the condition (traditionally referred to as a continuous observability inequality)

\[ \left\| L^*_T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_{H^{-k}(\Sigma_1) \times L_2(\Sigma_2)} \geq c_T \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_{Y_{1,\gamma} = D(\mathcal{A}) \times D(\mathcal{A}_\gamma^{1/2})}, \]  

where \( H^{-k}(\Sigma_1) = [H^0_0(\Sigma_1)]' \), for some constant \( c_T > 0 \), where the adjoint \( L^*_T \) of \( L_T \) in (5.1.1) is defined by the identity involving duality pairings:

\[ \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)_{Y_{1,\gamma}} = \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, L^*_T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)_{L_2(\Sigma_1) \times L_2(\Sigma_2)}, \]  

where \([u_1, u_2] \in H^0_0(\Sigma_1) \times L_2(\Sigma_2)\) and \( L^*_T x_1, x_2 \in H^{-k}(\Sigma_1) \times L_2(\Sigma_2)\).

**Lemma 5.1.** Let \( x = [\psi_0, \psi_1] \in Y_{1,\gamma} \). With reference to (5.1), (5.4), we have:

(i)

\[ \left( L^*_T \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \right)(t) = \begin{bmatrix} (1)(t) \\ (2)(t) \end{bmatrix}, \quad (2)(t) = -D^{+} \partial_x \psi_1(T - t; x) \]

\[ = \partial_x \psi_1(T - t; x), \]  

\[ (5.5) \]
for the (dual) $\psi$

Orientation $D$

where $\psi$

so that

rewritten as: there exists $\sim$

where

$\psi_1$

In this section we prove the continuous observability inequality (5.11)

Proof. See [E-L-T.2].

6. STEP 1. EXACT CONTROLLABILITY OF THE $\nu$-KIRCHHOFF

PROBLEM (3.2). THEOREM 4.1.2 BY DUALITY

Orientation

In this section we prove the continuous observability inequality (5.11) for the (dual) $\psi$-problem (3.6); thus, by duality between (5.2) and (5.3),
the equivalent exact controllability statement (5.2) for the \( v \)-problem (3.2) under boundary controls \( \{u_1, u_2\} \in H_0^k(\Sigma_1) \times L_2(\Sigma_2) \), any \( k \geq 2 \). This will be done by first establishing Proposition 6.1 below, which gives a related inequality polluted by (interior) lower-order terms, and then Proposition 6.2 below, which absorbs those (interior) lower-order terms in terms of the required traces. The combination of Propositions 6.1 and 6.2 will imply the continuous observability inequality (5.11) in Theorem 6.3, via the uniqueness result in [E-L-T .2, Corollary 8.2; or E-L-T .3].

**Proposition 6.1.** With reference to the \( \psi \)-dynamics in (5.8), that is, to the homogeneous problem (3.6), with initial conditions
\[
\begin{align*}
\{\psi(0), \psi_t(0)\} & = \{\psi_0, \psi_1\} \in Y_{1, \gamma} = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})^{1/2} \\
& = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)
\end{align*}
\] (see (2.3)), the following inequality holds true: there exists a constant \( c_T > 0 \) such that
\[
\begin{align*}
\int_0^T \int_{\Gamma} \left( \frac{\partial \psi_t}{\partial \nu} (T - t; x) \right)^2 d\Sigma + \left\| \{\psi, \psi_t\} \right\|_{L_2(0,T;L^2(\Omega) \times L_2(\Omega))}^2 & \geq c_T \left\| \{\psi_0, \psi_1\} \right\|_{[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)}^2, \\
& \quad \epsilon > 0.
\end{align*}
\] (6.2)

The boundary term \( \int_{\Gamma} \) over all of \( \Gamma \) can be replaced by \( \int_{\Gamma_2} \) only over \( \Gamma_2 \), if condition (4.1.4) (i.e., (1.3)) is assumed.

**Proof.**

Step (i). As in [L-T.11], we reduce the problem to the wave equation with variable coefficient \( \alpha^2(x) \) in the damping term. We then apply known estimates to this wave equation problem. To do this, we first recall the *a priori* regularity of the \( \psi \)-problem (3.6), or (5.8), given by Lemma 3.1(ii):
\[
\begin{align*}
\{\psi_0, \psi_1\} & \in Y_{1, \gamma} \quad \Rightarrow \\
\{\psi, \psi_t\} & \in C([0, T]; Y_{1, \gamma} = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)).
\end{align*}
\] (6.3)

We then note the following obvious identities (see (2.1a) for \( \mathcal{A}_\gamma \)):
\[
\begin{align*}
\mathcal{A}^{-1}_\gamma g & = \frac{g}{\gamma} - \frac{\mathcal{A}^{-1}_\gamma g}{\gamma}, \quad g \in L_2(\Omega), \quad \mathcal{A}_\gamma = (I + \gamma \mathcal{A}), \quad (6.4a) \\
\mathcal{A}^{-1}_\gamma \mathcal{A}^2 g & = \frac{1}{\gamma} \mathcal{A} g - \frac{g}{\gamma^2} + \frac{\mathcal{A}^{-1}_\gamma g}{\gamma^2}, \quad g \in \mathcal{D}(\mathcal{A}), \quad (6.4b)
\end{align*}
\]
\[ F_{ag} \equiv \text{div} (\alpha \nabla (ag)) = \text{div} \nabla (\alpha^2 g) - \text{div}(ag \nabla \alpha) \]
\[ = -\partial_t (\alpha^2 g) - \text{div}(ag \nabla \alpha), \quad g \in D(\partial_t), \quad (6.5a) \]
\[ \partial_t^{-1} F_{ag} = -\frac{\alpha^2 g}{\gamma} + \partial_t^{-1} (\alpha^2 g) \gamma - \partial_t^{-1} \text{div}(ag \nabla \alpha), \quad g \in L_2(\Omega). \quad (6.5b) \]

Hence, from (3.7), (6.3), and (6.5b),
\[ \psi_{tt} = -\frac{1}{\gamma} \partial_t \psi - \frac{\alpha^2}{\gamma} \psi_t + \ell \text{.o.t.} \quad (6.6a) \]
and from (6.4a), (6.4b), and (6.5b), we obtain
\[ \psi_{tt} = -\frac{1}{\gamma} \partial_t \psi - \frac{\alpha^2}{\gamma} \psi_t + \ell \text{.o.t.} \quad (6.6b) \]

where \( \ell \text{.o.t.} \) stands for lower-order terms. Next, we introduce a new variable \( z \),
\[ z(t) \equiv \psi_t(t; x) \in C([0, T]; H^1_0(\Omega)), \quad (6.7) \]
\[ z(0) = \psi_t(0; x) = \psi_1 \in H^1_0(\Omega), \]
\[ z_t(0) = \psi_{tt}(0; x) = -\partial_t^{-1} (\alpha \psi_0 + \partial_t^{-1} F_{ag} \psi_1) \in L_2(\Omega), \quad (6.8) \]
by (6.1) and (6.6). Next, differentiating (6.6) in \( t \), and using (6.7) yields, by (6.4b), (6.5b),
\[ z_{tt} = -\partial_t^{-1} (\alpha \partial_t z + \partial_t^{-1} F_{ag} z_t) = -\frac{1}{\gamma} \partial_t z - \frac{\alpha^2}{\gamma} z_t + f, \quad (6.9) \]
so that we obtain the following wave equation problem in \( z \),
\[ z_{tt} = \frac{\Delta z}{\gamma} - \frac{\alpha^2 z_t}{\gamma} + f \quad \text{in } Q; \quad (6.10a) \]
\[ z|_{\Sigma} \equiv 0 \quad \text{in } \Sigma, \quad (6.10b) \]
with variable coefficient \( \alpha^2(x) \) in front of the damping term \( z_t \),
\[ f \equiv \frac{z}{\gamma} - \partial_t^{-1} \text{div}(\alpha z \nabla \alpha) + \ell \text{.o.t.} = \frac{\psi_t}{\gamma^2} - \partial_t^{-1} \text{div}(\alpha \psi_0 \nabla \alpha) + \ell \text{.o.t.} \quad (6.11) \]

**Step (ii).** We invoke [L-T.6, Theorem 2.1.2(ii), Eq. (2.1.10b)] (which applies to a general wave equation with variable coefficients in the space variable in the first-order terms (energy level)). We obtain, in the notation of [L-T.6],
\[ (\bar{B} \bar{T})|_{\Sigma} + \frac{C_T}{\gamma^2} \int_{Q} f^2 dQ + TC_{T, \gamma} \|z\|_{L_2(\Omega)}^2 \geq k_{\gamma, T} E_z(0), \quad (6.12) \]
where in the equivalent norm via (6.8),
\[ E_\epsilon(0) = \| \{ z(0), z_t(0) \} \|_{\mathcal{D}(\partial^{1/2}) \times L_2(\Omega)} \]  
(6.13)

(the \( \mathcal{D}(\phi^{1/2}) \)-norm is equivalent to the \( H_0^1(\Omega) \)-norm for the \( z \)-variable), and where via [L-T6, (2.1.11) and (2.1.9)] we have by (6.10b) [so that \( h \cdot \nabla z = (\partial z/\partial \nu) h \cdot \nu, |\nabla z|^2 = (\partial z/\partial \nu)^2, h = (x - x_0) \)]

\[ (\overline{BT})_\Sigma = \int_\Sigma e^{it\phi} \frac{\partial z}{\partial \nu} h \cdot \nabla z d\Sigma - \frac{1}{2} \int_\Sigma e^{it\phi} |\nabla z|^2 h \cdot \nu d\Sigma \]
\[ = \frac{1}{2} \int_\Sigma e^{it\phi} \left( \frac{\partial z}{\partial \nu} \right)^2 h \cdot \nu d\Sigma \leq C_\phi \int_0^T \int_{\Gamma_2} \left( \frac{\partial z}{\partial \nu} \right)^2 d\Sigma_2 \]  
(6.14)

(\( \phi \) is the pseudo-convex function in [L-T6]). In the last step in (6.15), we have invoked assumption (4.1.4) on \( \Gamma_2 \).

Step (iii). From the definition of \( f \) in (6.11), we shall obtain, with \( \alpha \in C^1(\overline{\Omega}) \), that

\[ \| f \|_{L_2(\Omega)}^2 \leq C_\alpha \left\{ \| \psi \|_{L_2(L^{3/2+\epsilon}(\Omega))}^2 + \| \psi_t \|_{L_2(L^{3/2+\epsilon}(\Omega))} \right\}, \quad \epsilon > 0. \]  
(6.16)

Indeed, by (6.11), it suffices to show that

\[ \| \mathcal{D}^{-1} \mathcal{L} \|_{L_2(\Omega)} \]
\[ \leq C_\alpha \left\{ \| \psi \|_{L_2(L^{3/2+\epsilon}(\Omega))}^2 + \| \psi_t \|_{L_2(L^{3/2+\epsilon}(\Omega))} \right\}. \]  
(6.17)

To this end, we recall that for the operator \( \mathcal{D} \) in (2.1a) we have [L-T10] for any \( \epsilon > 0 \),

\[ \mathcal{D}(\partial^{3/4-\epsilon}) = H^{3/2-2\epsilon}(\Omega), \quad \text{that } \left\{ \mathcal{D}(\partial^{3/4-\epsilon}) \right\}' = H^{-3/2+2\epsilon}(\Omega). \]  
(6.18)

Thus, returning to (6.6b) and using (6.18), we have

\[ \| \mathcal{D}^{-1} \mathcal{L} \|_{L_2(\Omega)} = \| \mathcal{L} \|_{[\mathcal{D}(\partial^{3/4-\epsilon})]' \to [\mathcal{D}(\partial^{3/4-\epsilon})]} \]
\[ \leq \| \mathcal{L} \|_{L^{3/2+\epsilon}(\Omega)} \]
\[ \leq C_\alpha \| \psi \|_{L^{3/2+\epsilon}(\Omega)} \]  
(6.19)

(by (6.6b))
\[ \leq \frac{C_\alpha}{\gamma} \left\{ \| \psi \|_{L^{3/2+\epsilon}(\Omega)} + \| \psi_t \|_{L^{3/2+\epsilon}(\Omega)} \right\} \]
\[ \leq \frac{C_\alpha}{\gamma} \left\{ \| \psi \|_{L^{3/2+\epsilon}(\Omega)} + \| \psi_t \|_{L^{3/2+\epsilon}(\Omega)} \right\}. \]  
(6.20)

Then, (6.20) proves a fortiori (6.17), and this, in turn, establishes (6.16).
Step (iv). Using estimate (6.16) in (6.12) yields the desired estimate (6.2), by recalling (6.15), (6.13), (6.8) and \( z = \psi_t \) by (6.7). Proposition 6.1 is proved.

We note that the term \( \{ \psi, \psi_t \} \) in (6.2) is a lower-order term with respect to the \( H^2(\Omega) \times H^1(\Omega) \)-energy level in (6.3). We shall now absorb it by the desired traces.

**Proposition 6.2.** Let \( T > T_{0, b} \); see (4.1.5). Let \( \psi \) be a solution of (3.6), i.e., of the form (5.8), thus satisfying estimate (6.2). Then, the following inequality holds true: there exists a constant \( C_T > 0 \), \( C_T = C_{T, \gamma, k} \) (\( \gamma > 0 \) fixed), such that

\[
C_T \left\| \psi, \psi_t \right\|_{L_2(0,T;H^0_0(\Omega) \times L_2(\Omega))}^2 
\leq \left\| \frac{\partial \psi_t}{\partial \nu} \right\|_{L_2(\Sigma_2)}^2 + \left\| \frac{\partial \Delta \psi_t}{\partial \nu} \right\|_{H^{-1}(\Sigma_1)}^2. \tag{6.21}
\]

**Proof.** By contradiction, suppose that inequality (6.21) is false. Then, there exists a sequence

\[
\{ \psi^{(n)}, \psi_t^{(n)} \} \subset C([0, T]; \mathscr{D}(\partial) \times \mathscr{D}(\partial^{1/2}))
\]

continuous in the initial data \( \{ \psi^{(n)}_0, \psi_t^{(n)} \} \in \mathscr{D}(\partial) \times \mathscr{D}(\partial^{1/2}) \) (6.22) of solutions to problem (3.6) (hinged B.C.), i.e., of the form (5.8), such that

\[
\| \{ \psi^{(n)}, \psi_t^{(n)} \} \|_{L_2(0,T;H^0_0(\Omega) \times L_2(\Omega))} \equiv 1, \tag{6.23}
\]

\[
\left\| \frac{\partial \psi_t^{(n)}}{\partial \nu} \right\|_{L_2(\Sigma_2)}^2 + \left\| \frac{\partial \Delta \psi_t^{(n)}}{\partial \nu} \right\|_{H^{-1}(\Sigma_1)}^2 \to 0 \text{ as } n \to \infty. \tag{6.24}
\]

The sequence \( \{ \psi^{(n)} \} \) satisfies inequality (6.2). Thus, by (6.23), (6.24), the corresponding initial conditions are uniformly bounded. Then, by (6.22) and by (6.6) applied to \( \psi^{(n)} \), we have via (6.4b), (6.5b), and (6.26):

\[
\| \{ \psi^{(n)}, \psi_t^{(n)} \} \|_{C([0,T];H^0_0(\Omega) \times H^1(\Omega))} + \| \psi_t^{(n)} \|_{C([0,T];L_2(\Omega))} \leq C, \quad \forall n. \tag{6.25}
\]

Then, *a fortiori* via (6.25), we can apply Aubin’s lemma [A.1], since \( \mathscr{A}^{-1} \) is compact on \( L_2(\Omega) \), and obtain for a subsequence

\[
\{ \psi^{(n)}, \psi_t^{(n)} \} \to \text{ some } \{ \tilde{\psi}, \tilde{\psi}_t \} \text{ strongly in } L_2(0,T;H^0_0(\Omega) \times L_2(\Omega)). \tag{6.26}
\]

Then (6.23) and (6.26) yield

\[
\| \{ \tilde{\psi}, \tilde{\psi}_t \} \|_{L_2(0,T;H^0_0(\Omega) \times L_2(\Omega))} = 1. \tag{6.27}
\]
The limit $\tilde{\psi}$ satisfies problem (3.6), in particular, the BC

$$\tilde{\psi}|_{\Sigma} \equiv 0 \quad \text{and} \quad \Delta \tilde{\psi}|_{\Sigma} \equiv 0,$$

hence $\tilde{\psi}_t|_{\Sigma} \equiv 0$ and $\Delta \tilde{\psi}_t|_{\Sigma} \equiv 0$. \hspace{1em} (6.28)

Moreover, by (6.24),

$$\left. \frac{\partial \tilde{\psi}_t}{\partial n} \right|_{\Sigma_2} \equiv 0 \quad \text{on} \quad \Sigma_2 \quad \text{and} \quad \left. \frac{\partial \Delta \tilde{\psi}_t}{\partial n} \right|_{\Sigma_1} \equiv 0 \quad \text{on} \quad \Sigma_1. \hspace{1em} (6.29)$$

Moreover, $\tilde{\psi}_t$ itself satisfies the Kirchoff equation (3.6a) (by differentiating in $t$). Thus, $\tilde{\psi}$ satisfies the Kirchoff equation (3.6a) with over-determined homogeneous BC (6.28) and (6.29).

Recalling that $T > T_{0, h}$, we can invoke [E-L-T.2, Corollary 8.2, Section 8; E-L-T.3], and conclude that, in fact, $\tilde{\psi}_t \equiv 0$ in $Q$, and $\tilde{\psi} \equiv$ const in $Q$. But $\tilde{\psi}|_{\Sigma} \equiv 0$ by (6.28), and thus $\tilde{\psi} \equiv 0$ in $Q$. But this contradicts (6.27). Thus, inequality (6.21) holds true.

**THEOREM 6.3.** Let $T > T_{0, h}$; see (4.1.5). With reference to the dynamics in (5.8), that is, to the homogeneous problem (3.6) (hinged case) with initial conditions as in (6.1), we have that the continuous observability inequality (5.11) holds true.

**Proof.** We combine (6.2) of Proposition 6.1 with (6.21) of Proposition 6.2 to obtain (5.11).

**REFERENCES**


I. Lasiecka and R. Triggiani, “Differential and Algebraic Riccati Equations with


